UNIFORMITY OF UNIFORM CONVERGENCE ON THE FAMILY OF SETS

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Abstract. We prove that for every Hausdorff space X and any uniform quadra space $(Y,\mathcal{U})$ the topology on $C(X,Y)$ induced by the uniformity $\hat{\mathcal{U}}|\lambda$ of uniform convergence on the saturation family λ coincides with the set-open topology on $C(X,Y)$. In particular, for every pseudocompact space X and any metrizable topological vector space Y with uniform $\mathcal{U}$ the topology on $C(X,Y)$ induced by the uniformity $\hat{\mathcal{U}}$ of uniform convergence coincides with the C-compact-open topology on $C(X,Y)$, and depends only on the topology induced on Y by the uniformity $\mathcal{U}$. It is also shown that in the class closed-homogeneous complete uniform spaces Y necessary condition for coincidence of topologies is Y-compactness of elements of family λ.

1. Introduction

Let X be a Hausdorff space and let $(Y,\mathcal{U})$ be a uniform space. We shall denote by $C(X,Y)$ the set of all continuous mappings of the space X to the space Y, where Y is equipped with the topology induced by $\mathcal{U}$. For every $V\in\mathcal{U}$ denote by $\hat{V}$ the entourage of the diagonal $\Delta\subset C(X,Y)\times C(X,Y)$ defined by the formula

$\hat{V}=\{(f,g):(f(x),g(x))\in V$ for every $x\in X\}$.

The uniformity on the set $C(X,Y)$ generated by this family is called the uniformity of uniform convergence induced by $\mathcal{U}$ and will be denoted $\hat{\mathcal{U}}$. For two uniformities $\mathcal{U}_1$ and $\mathcal{U}_2$ on Y which induce the same topology, the topologies on $C(X,Y)$ induced by $\hat{\mathcal{U}}_1$ and $\hat{\mathcal{U}}_2$

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can be different (example 4.2.14 in [3]). It turns out, however, that for a compact space X the topology on $C(X,Y)$ is independent of the choice of a particular uniformity $\mathcal{U}$ on the space Y, because the topology induced by $\hat{\mathcal{U}}$ coincides with the compact-open topology on $C(X,Y)$.

2. Preliminaries

Let X and Y be topological spaces. For a fixed natural number n, a subset A of X is said to be Y^n-*compact* provided $f(A)$ is a compact for any $f \in C(X,Y^n)$.

For example, if space Y is a metrizable topological vector space then a Y-compact subset A of X is a C-compact subset of X and, moreover, A is a Y^n-compact subset of X for any $n \in \mathbb{N}$ (and even Y^ω-compact) [5]. Recall that a subset A of space X is a C-compact subset of X provided that a set $f(A)$ is compact for every $f \in C(X,\mathbb{R})$. Note that any Y^{n+1}-compact subset of X is a Y^n-compact subset of X.

Definition 2.1. A space Y is called *quadra* space if for any $x \in Y \times Y$ there are a continuous map f from $Y \times Y$ to Y and a point $y \in Y$ such that $f^{-1}(y) = x$.

For example, any space with G_δ-diagonal containing nontrivial path or a zero-dimensional space with G_δ-diagonal is a quadra space. In [2] a space M_1 with the following properties is constructed: M_1 is a metric continuum; if Z is a sub-continuum of M_1, $f : Z \mapsto M_1$ is a continuous mapping, then either f is constant or $f(x) = x$ for all $x \in X$. It follows that M_1 is not a quadra space.

Proposition 2.2. *Let Y be a quadra space. Then any Y-compact subset of X is Y^2-compact subset of X.*

Proof. Let A is a Y-compact subset X and $g \in C(X, Y \times Y)$. Suppose that there is $z \in \overline{g(A)} \setminus g(A)$. So there are a continuous map f from $Y \times Y$ to Y and a point $y \in Y$ such that $f^{-1}(y) = z$. It follows that $f(g(A))$ is not compact subset of Y which contradicts the Y-compactness of A. $\square$

A subset A of X is said to be *Y-zero-set* provided $A = f^{-1}(Z)$ for some zero-set Z of Y and $f \in C(X, Y)$. For example, if space Y is real numbers $\mathbb{R}$ then any zero-set subset of X is a $\mathbb{R}$-zero-set of X.

Proposition 2.3. *Let X and Y be topological spaces, A be a Y^2-compact subset of X and B be a Y-zero-set such that $B \bigcap A \neq \emptyset$. Then $B \bigcap A$ is Y-compact subset of X.*

Proof. Let $g \in C(X, Y)$. We fix a continuous mapping h of Y into $\mathbb{R}$ such that $Z = h^{-1}(0)$. Let $f \in C(X, Y)$ such that $B = f^{-1}(Z)$. Let f_1 be the diagonal product of the mappings g and f, that is, $f_1(x) = (g(x), f(x)) \in Y \times Y$. The set $S = f_1(B \bigcap A) = f_1(A) \bigcap (Y \times Z)$ is closed in $Y \times Y$, and it follows that S is compact.

Let π be natural projection of $Y \times Y$ onto Y, associating with every point its first coordinate. Then, clearly, $g = \pi \circ f_1$ and $g(B \bigcap A) = \pi(S)$.

Since π is continuous and S is compact, we conclude that $g(B \bigcap A)$ is also compact.

□

Proposition 2.4. *Let X be a topological space, Y be a quadra space, A be a Y-compact subset of X and B be a Y-zero-set such that $B \bigcap A \neq \emptyset$. Then $B \bigcap A$ is Y-compact subset of X.*

3. Uniformity of uniform convergence on Y-compact sets

Recall that a family λ of non-empty subsets of a topological space (X, τ) is called a π-network for X if for any nonempty open set $U \in \tau$ there exists $A \in \lambda$ such that $A \subseteq U$.

For a Hausdorff space X, a π-network λ for X and a uniform space $(Y, \mathcal{U})$ we shall denote by $\hat{\mathcal{U}}|\lambda$ the uniformity on $C(X, Y)$ generated by the base consisting of all finite intersections of the sets of the form

$\hat{V}|A = \{(f, g) : (f(x), g(x)) \in V$ for every $x \in A\}$, where $V \in \mathcal{U}$, $A \in \lambda$.

The uniformity $\hat{\mathcal{U}}|\lambda$ will be called the uniformity of uniform convergence on family λ induced by $\mathcal{U}$.

Recall that all sets of the form $\{f \in C(X,Y) : \ f(F) \subseteq U\}$, where $F \in \lambda$ and U is an open subset of Y, form a subbase of the set-open (λ-open) topology on $C(X,Y)$.

We use the following notations for various topological spaces on the set $C(X,Y)$:

$C_{\hat{\mathcal{U}}|\lambda}(X,Y)$ for the topology induced by $\hat{\mathcal{U}}|\lambda$,

$C_{\lambda}(X,Y)$ for the λ-open topology.

Let y be a point of a uniform space $(Y,\mathcal{U})$ and let $V \in \mathcal{U}$. Recall that the set $B(y,V) = \{z \in Y : (y,z) \in V\}$ is called the ball with centre y and radius V or, briefly, the V-ball about y. For a set $A \subset Y$ and a $V \in \mathcal{U}$, by the V-ball about A we mean the set $B(A,V) = \bigcup\limits_{y \in A} B(y,V)$.

Lemma 3.1. *(Lemma 8.2.5. in [3]). If $\mathcal{U}$ is a uniformity on a space X, then for every compact set $Z \subset X$ and any open set G containing Z there exists a $V \in \mathcal{U}$ such that $B(Z,V) \subset G$.*

A family λ will be called hereditary with respect to Y-zero-set subsets of X if any nonempty $A \bigcap B \in \lambda$ where $A \in \lambda$ and B is a Y-zero-set of X.

Definition 3.2. Let X be Hausdorff space and let $(Y,\mathcal{U})$ be a uniform quadra space. Let a family λ of Y-compact subsets of X be π-network for X and it hereditary with respect to Y-zero-set subsets X, then we say that λ is *saturation* family.

Theorem 3.3. *For every Hausdorff space X and any uniform quadra space $(Y,\mathcal{U})$ the topology on $C(X,Y)$ induced by the uniformity $\hat{\mathcal{U}}|\lambda$ of uniform convergence on the saturation family λ coincides with the λ-open topology on $C(X,Y)$, where Y has the topology induced by $\mathcal{U}$.*

Proof. Denote by τ_1 the topology on $C(X,Y)$ induced by the uniformity $\hat{\mathcal{U}}|\lambda$ and by τ_2 the λ-open topology. First we shall prove that $\tau_2 \subseteq \tau_1$. Clearly, it suffices to show that all sets $[A,U]$, where $A \in \lambda$ and U is an open subset of Y, belong to τ_1. Consider a $A \in \lambda$, an open set $U \subseteq Y$ and an $f \in [A,U]$. Since A is a Y-compact subset of X, $f(A)$ is a compact subspace of U. Applying Lemma 3.1., take a $V \in \mathcal{U}$ such that $B(f(A),V) \subseteq U$. We have

$B(f, \hat{V}|A) \subseteq [A, U]$, and f being an arbitrary element of $[A, U]$, this implies that $[A, U] \in \tau_1$.

We shall now prove that $\tau_1 \subseteq \tau_2$. Clearly, it suffices to show that for any $A \in \lambda$, $V \in \mathcal{U}$ and $f \in C(X, Y)$ there exist Y-compact subsets A_1,...,$A_k \in \lambda$ and open subsets $U_1, ..., U_k$ of Y such that

$f \in \bigcap\limits_{i=1}^{k} [A_i, U_i] \subset B(f, \hat{V}|A)$.

By Corollary 8.1.12 in [3] there exists an entourage $W \in \mathcal{U}$ of the diagonal $\Delta \subset Y \times Y$ which is closed with respect to the topology induced by $\mathcal{U}$ on $Y \times Y$ and satisfies the inclusion $3W \subset V$. It follows from the compactness of $f(A)$ that there exists a finite set $\{x_1, ..., x_k\} \subset A$ such that $f(A) \subseteq \bigcup_{i=1}^{k} B(f(x_i, W)$. Note that $f(A) \subset \bigcup\limits_{i=1}^{k} U_i$ where $U_i = IntB(f(x_i), 2W)$. Observe that from the closedness of W in $Y \times Y$ follows the closedness of balls $B(f(x_i), W)$ in Y and the compactness of the sets $f(A) \bigcap B(f(x_i), W)$. Let Z_i be a zero-sets of Y such that $f(A) \bigcap B(f(x_i), W) \subseteq Z_i \subseteq U_i$. By the Proposition 2.4, sets $A_i = f^{-1}(Z_i) \bigcap A$ is Y-compact subsets. Note that $A_i \in \lambda$ because the family is saturation family.

We have $f \in \bigcap\limits_{i=1}^{k} [A_i, U_i]$. If $g \in \bigcap\limits_{i=1}^{k} [A_i, U_i]$ then for any $x \in A$ there is A_i such that $x \in A_i$ and we have $g(x) \in B(f(x_i), 2W)$ and $f(x) \in B(f(x_i), W)$. It follows that $(f(x), g(x)) \in 3W \subset V$ for any $x \in A$ and $g \in B(f, \hat{V}|A)$.

□

The Y-compact-open topology on $C(X, Y)$ is the topology generated by the base consisting of sets $\bigcap\limits_{i=1}^{k} [A_i, U_i]$, where A_i is a Y-compact subset of X and U_i is an open subset of Y for $i = 1, ..., k$.

Corollary 3.4. For every Y^2-compact space X and any uniform space $(Y,\mathcal{U})$ the topology on $C(X, Y)$ induced by the uniformity $\hat{\mathcal{U}}$ of uniform convergence coincides with the Y-compact-open topology on $C(X, Y)$, and depends only on the topology induced on Y by the uniformity $\mathcal{U}$.

Note that $\mathbb{R}$-compactness (C-compactness) of a space X is pseudocompactness of X.

Corollary 3.5. For every pseudocompact space X and any metrizable topological vector space Y with uniform $\mathcal{U}$ the topology on $C(X,Y)$ induced by the uniformity $\hat{\mathcal{U}}$ of uniform convergence coincides with the C-compact-open topology on $C(X,Y)$, and depends only on the topology induced on Y by the uniformity $\mathcal{U}$.

4. Closed-homogeneous spaces

Recall that a space X is strongly locally homogeneous (abbreviated: SLH) if it has an open base $\mathcal{B}$ such that for all $B \in \mathcal{B}$ and $x, y \in B$ there is a homeomorphism $f : X \mapsto X$ which is supported on B (that is, f is the identity outside B) and moves x to y. The well-known homogeneous continua are strongly locally homogeneous: the Hilbert cube, the universal Menger continua and manifolds without boundaries.

A topological space X is said to be closed-homogeneous provided that for any $x, y \in X$ and any K closed subset of $X \setminus \{x, y\}$, there is a homeomorphism $f : X \mapsto X$ which is supported on $X \setminus K$ (that is, f is the identity on K) and moves x to y.

The well-known that a zero-dimensional homogeneous space is closed-homogeneous. Observe that a closed-homogeneous space is SLH. Note that there exists an SLH space X which is not closed-homogeneous (see [4]). In fact, if we take $X = \mathbb{R} \setminus \{0\}$, $\beta = \{\{x\} : x < 0\} \bigcup \{(a, b) : 0 < a < b\}$. Then the topological space $(X, \tau(\beta))$; generated by the base β; is an SLH metrizable space which is not closed-homogeneous.

5. Uniformity of uniform convergence on Y-closed totally bounded sets

Recall that if Y is uniformized by a uniformity $\hat{U}$, a subset A of X is said Y-totally bounded when $f(A)$ is totally bounded for any $f \in C(X,Y)$ (see [1]).

A subset A of X is said to be Y-*closed totally bounded* if $f(A)$ is closed totally bounded for any $f \in C(X,Y)$.

Theorem 5.1. *Let X be a Hausdorff space, Y be uniform closed-homogeneous space and $C_{\hat{\mathcal{U}}|\lambda}(X,Y) = C_{\lambda}(X,Y)$. Then, the family λ consists of Y-closed totally bounded sets.*

Proof. Suppose that there is $A \in \lambda$ which is not Y-totally bounded set. Then, there is $f \in C(X,Y)$ such that $f(A)$ is not totally bounded. Let $B(f, \hat{V}|A)$ be an open neighborhood of f in the topological space $C_{\hat{\mathcal{U}}|\lambda}(X,Y)$.

Since $C_{\hat{\mathcal{U}}|\lambda}(X,Y) = C_{\lambda}(X,Y)$, there is an open set $\bigcap\limits_{i=1}^{k} [A_i, U_i]$ in the topological space $C_{\lambda}(X,Y)$ such that $f \in \bigcap\limits_{i=1}^{k} [A_i, U_i] \subseteq B(f, \hat{V}|A)$. Consider a subset M of $f(A)$ such that:

1. M is not totally bounded;
2. either $M \subset U_i$ or $\overline{f(A) \bigcap U_i} \bigcap M = \emptyset$ for every $i = 1, ..., k$.

Let $W = \bigcap U_i$ where U_i such that $M \subseteq U_i$. Let $y_1, y_2 \in W$ such that $y_1 \in M$ and $(y_1, y_2) \notin V$. Since Y is a closed-homogeneous space there is a homeomorphism $h : Y \mapsto Y$ which is supported on W (that is, h is the identity on $X \setminus W$) and moves y_1 to y_2. Consider a continuous map $g = h \circ f$. Note that $g \in \bigcap\limits_{i=1}^{k} [A_i, U_i]$. It is clear that if $x \in f^{-1}(y_1) \bigcap A$ then $(f(x), g(x)) \notin V$ and $g \notin B(f, \hat{V}|A)$. This contradicts our assumption. So a set $f(A)$ is a totally bounded subset of space Y and A is a Y-totally bounded set.

Suppose that $f(A)$ is not closed. Then we have a point $y \in \overline{f(A)} \setminus f(A)$. Let $S = Y \setminus \{y\}$ and $[A, S]$ be an open set of space $C_{\lambda}(X,Y)$. Then there exists an open set $B(f, \hat{V}|B)$ of space $C_{\hat{\mathcal{U}}|\lambda}(X,Y)$ such that $f \in B(f, \hat{V}|B) \subseteq [A, S]$. Let z be a point of $IntB(y, W)$ where $2W \subseteq V$ such that $f^{-1}(z) \bigcap A \neq \emptyset$. Since Y is a closed-homogeneous space there is a homeomorphism $p : Y \mapsto Y$ which is supported on $IntB(y, W)$ and moves z to y. Consider a continuous map $q = p \circ f$. It is clear that if $x \in f^{-1}(IntB(y, W)) \bigcap B$ then $(f(x), q(x)) \in 2W \subseteq V$ and if $x \in f^{-1}(z) \bigcap A$ then $q(x) = y$. Thus $q \in B(f, \hat{V}|B)$ and $q \notin [A, S]$. This contradicts our assumption. We have that a set A is a Y-closed totally compact subset of a space X.

□

Theorem 5.2. *Let X be a Hausdorff space, Y be closed-homogeneous complete uniform space and $C_{\hat{\mathcal{U}}|\lambda}(X,Y) = C_{\lambda}(X,Y)$. Then, the family λ consists of Y-compact sets.*

Proof. It suffices to note that a closed totally bounded subset of complete uniform space is a compact set.

□

Corollary 5.3. Let X be a Hausdorff space, Y be zero-dimensional homogeneous complete uniform space and $C_{\hat{\mathcal{U}}|\lambda}(X,Y) = C_\lambda(X,Y)$. Then, the family λ consists of Y-compact sets.

Example 5.4. *If Z is the Sorgenfrey line and $C_{\hat{\mathcal{U}}|\lambda}(Z,Z) = C_\lambda(Z,Z)$ then the family λ consists of compact sets. Since Z is a quadra space then we get that for any Hausdorff space X the topology on $C(X,Z)$ induced by the uniformity $\hat{\mathcal{U}}|\lambda$ of uniform convergence on the saturation compact family λ coincides with the λ-open topology on $C(X,Z)$.*

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